Victor Aguilar English 321 4:00 February 27, 1991

<u>Intended audience</u>: Any high school graduate has sufficient background to understand this essay. Nevertheless, the arguments presented here are carried out farther than is typical of a high school discussion and the essay will probably not be susceptible to a cursory reading.

<u>Purpose</u>: To introduce an interesting problem and to discuss the methodology taken in solving it.

Essay on traveling farther than the range of one's vehicle

Consider the following problem: A man wishes to drive 100 miles in a vehicle with a range of only 50 miles. Obviously, he will have to carry fuel out to be recovered later. But where should he place the checkpoints and how much fuel should he leave at them to minimize travel time? Notice that this is not necessarily the same as minimizing fuel consumption since it is not at all obvious that a method can be designed which avoids leaving any fuel along the way or that avoidance of this is particularly important. The most likely scenario would be escaping from a foreign country on a snowmobile. There is a definite limit on the amount of fuel one can carry without sinking in soft powder and, if it's relatively flat ground, local variances in gas mileage would be negligible. Since the chance of being strafed by a passing airplane is proportional to travel time, one would like to give some thought to its minimization.

Since it is initially unknown how much fuel needs to be taken from the starting point, the problem must be worked backwards.

The last leg of the trip from the 50 mile mark to the finish must

be driven straight through for maximum economy. Thus, the problem is reduced to carrying 1 tank of fuel out to the 50 mile mark. If the remaining distance is divided into three parts, the problem can be successively reduced to carrying a quantity of fuel out to each checkpoint which is sufficient for what will be consumed on the next leg plus what is needed thereafter. For instance, it takes 1 and 1/2 round trips to carry 1 tank from the 33.33 mile mark out to the 50 mile mark (1/3 tanks delivered on the first trip and 2/3 tanks delivered on the next trip). These trips consume 1 tank, so the problem is reduced to carrying 2 tanks out to the 33.33 mile mark. This takes 4 and 1/2 round trips from the 16.66 mile mark which consumes 3 tanks. Carrying 5 tanks out to the 16.66 mile mark requires 13 and 1/2 round trips which consumes 9 tanks for a total consumption of 14 tanks.

Superficially, this seems to be a minimum since no fuel has been left along the way. However, it is natural to inquire whether it would be more economical to take shorter trips and deliver larger quantities of fuel to each checkpoint. Dividing the first 50 miles by successively larger integers up to 20, one finds that with 15 segments only 7.8667 tanks of fuel are required and that, of the first 200 integers, 126 segments is the best at only 7.6984 tanks of fuel. This is considerably less than 14 tanks. The fuel consumption for a division into i segments, g(i), can be determined with the following program, the results of which can be sorted and the smallest chosen.

It is apparent that, despite the fact that many of the final one-way trips on certain legs have the capacity to deliver more fuel than is needed (the extra will be left behind when one abandons that checkpoint), the shorter trips with larger deliveries are more economical. If economy were a monotonic (strictly decreasing) function of the number of segments, then one would conclude that travel time asymptotically approaches its minimum as the number of segments tends toward infinity. not monotonic, however. In fact, it fluctuates considerably making one doubt that any particular finite integer can be identified with the minimum. The reason for the fluctuations is obvious: No attention has been given to the amount of fuel left along the way. Some of the segment's final one-way trips transport very little fuel and, with a slight change in segment length, one could eliminate an entire round trip. particularly important for the later segments since the fuel consumed on each round trip there must be transported over many previous segments at considerable cost. Hence, while the actual quantity of fuel left behind may seem negligible (and indeed can be eliminated by carrying less than one is able to at times), its effect on fuel economy is greatly amplified.

The question naturally arises, what is the largest integer divisor such that no fuel is left along the way? This is equivalent to requiring that the number of whole round trips, t, always be an integer without having to be rounded up. Consider the first time through the inner loop while g(i) still equals 1 and cancels the -1 so t = 1/(i-2). Obviously, i must equal 3 for t to be an integer. Thus, for larger i, some or even most of the legs on the trip will not leave fuel behind but at least one, the last, must. Therefore, to avoid leaving fuel along the way, one must relax the (tacit) condition that each segment evenly divides the range, that is, segment lengths need no longer be members of the harmonic progression r, r/2, r/3,...

To determine the shortest possible segments such that no fuel is abandoned, replace 1/i with f in the expression for t and, solving for f, run through the non-negative integers, t, to find the one with the smallest positive f. Each segment is then fr long, the distance a function only of the fuel needed at the end of that segment, g. One repeats this process until the sum of the segments (counting backwards from the total distance) reach the starting point. This can be calculated with the following program (whose commands to the user are printed in reverse order of their execution):

```
n = distance, 100 miles
r = range, 50 miles
do
  t = -1
  do
    t = t + 1
    f = (t - g + 1)/(2t + 1)
  until
    f > 0
  end
  t = t + 1/2
  "Make t round trips from n-fr to n miles dropping off 1-2f
   tanks on each whole round trip."
  n = n - fr
  g = g + 2ft
until
  n = < 0
end
```

If, after some number of steps, the distance is exactly zero, then one has provably minimized both travel time and fuel consumption. If n < 0 after running the above program, then the last step calculated (the first segment driven) can be shortened by n miles and the final one-way trip on that segment will have the capacity to carry more fuel than is needed. Having more capacity than needed on the first segment cannot be prevented. However, being the first segment, its effect on fuel economy is not amplified (as previously discussed) and, being the shortest segment, it has the least effect before amplification on fuel economy. Hence, with this adjustment on the length of the first segment having been made, the above listed program provides provably the most economical method of traveling farther than the range of one's vehicle. For n = 100 and r = 50, only 7.6730 tanks are consumed. The instructions to the driver are:

round trips	$\underline{\mathtt{from}}$	<u>to</u>	dropping off
7.5 6.5 5.5 4.5 3.5 2.5	0 2.2433 6.0895 10.6349 16.1905 23.3333	2.2433 6.0895 10.6349 16.1905 23.3333 33.3333	.9103 .8462 .8182 .7778 .7143
1.5 0.5	33.3333 50.0000 (miles)	50.0000 100.0000 (miles)	.3333 (tanks)

Notice that the number of round trips at each step is an arithmetic progression. This suggests that the program could have been written without a conditional test for each nonnegative integer. However, a more general case would be where one is transporting some quantity of fuel to the destination. The program as it is written can accommodate this situation simply by including g = amount transported among the inputs.

The greatest discrepancy with reality is the assumption that one's vehicle has a fixed range regardless of the load it carries. More realistically, the vehicle's instantaneous fuel economy would be a (linear) function of the amount of fuel in its tank at the moment. This is because the weight of the fuel in an off road vehicle's tank is the largest part of the load on its engine. Incorporating this information into the problem is, however, beyond the scope of the present essay.